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THE STABILITY AND SMALL OSCILLATIONS OF AN UNDULOIDAL FLUID BRIDGE BETWEEN TWO COAXIAL DISCS UNDER CONDITIONS OF WEIGHTLESSNESS†

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A fluid bridge between two identical coaxial discs is considered which, in equilibrium, has the form of a convex unduloid (that is, a wave-like surface). It is shown that the stability of the equilibrium and the existence of small oscillations of the fluid depend on the coercivity of the bilinear form associated with the operator arising in the problem which is determined by the potential of the surface tension forces. The problem reduces to an operator equation in which one of the operators is associated, by virtue of Laplace's law, with the mean curvature of the perturbed free surface. The problem of coercivity reduces to an auxiliary eigenvalue problem. The conditions of stability are found to be satisfied if all of the eigenvalues of the problem are strictly greater than unity. Sufficient conditions for stability are obtained using arguments based on the theory of elliptic functions. The existence of natural frequencies is proved using functional analysis methods.

The problem of the stability and small oscillations of a fluid mass under low gravity conditions, when surface tension plays a decisive role, has been the object of numerous investigations ([1–4] and others) including the case of a fluid bridge between two coaxial discs [5–7].

It is well known that, if the fluid mass has an axially symmetric form in equilibrium, its free surface, the mean curvature of which is constant, can be a cylinder, a sphere, an unduloid, a nodoid or a catenoid. In other words, this is a surface of revolution, the meridian of which is the geometric location of the focus of an ellipse, hyperbola or parabola that rolls without slipping along a certain straight line [8]. The problem of the stability and small oscillations of catenoidal and spherical bridges has been investigated in detail in [9, 10].

1. FORMULATION OF THE PROBLEM AND EQUATIONS OF MOTION

Consider a fluid bridge which occupies the domain τ between two coaxial discs S_1 and S_2, the boundaries of which C_1 and C_2 are two identical circles. We will assume that the free surface of the fluid in equilibrium is a convex wave-like surface (an unduloid) The equations of the surfaces S_1 and S_2 are given as $z = h$ and $z = -h$ respectively. We introduce a cylindrical system of coordinates r, θ, z. The equations [8]

$$r = a\cos\varphi + a\sqrt{e^2 - \sin^2\varphi}$$
$$z = a\sin\varphi + \operatorname{tg}\varphi\sqrt{e^2 - \sin^2\varphi} - \int_0^{\varphi} \frac{a(e^2 - 1)}{\cos^2 u\sqrt{e^2 - \sin^2 u}}du$$

determine the meridian of the convex unduloid which corresponds to an ellipse with semi major axis a and eccentricity e, $0 < e < 1$. The part of it located between the planes $z = h$ and $z = -h$ is described by the inequality

$$-\varphi_0 < -\varphi_1 \le \varphi \le \varphi_1 < \varphi_0; \quad \varphi_0 = \arcsin e, \quad 0 < \varphi_0 < \pi/2$$

where φ is the angle formed by the radius vector and the unit vector of the outward normal $\mathbf{n}_0$ to the surface S_0 and outward with respect to the fluid (Fig. 1).

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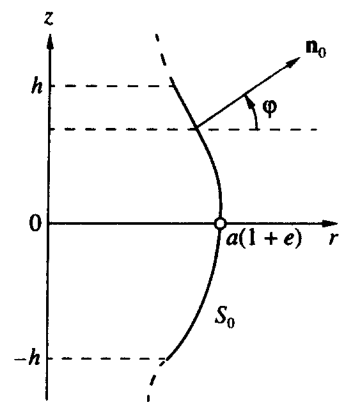

Fig. 1

It is natural to introduce elliptic functions. On putting

$$x = \int_0^{\varphi} \frac{du}{\sqrt{e^2 - \sin^2 u}}$$

as is customary [11], we have within the framework of the standard notation for Jacobian functions

$$\sin\varphi = e\,\mathrm{sn}\,x, \quad \cos\varphi = \mathrm{dn}\,x, \quad \sqrt{e^2 - \sin^2\varphi} = e\,\mathrm{cn}\,x, \quad d\varphi = e\,\mathrm{cn}\,x dx$$

We now introduce the notation

$$\chi_0(x) = (\mathrm{dn}\,x + e\,\mathrm{sn}\,x)^2, \quad \chi_1(x) = \mathrm{dn}\,x(\mathrm{dn}\,x + e\,\mathrm{sn}\,x)^2, \quad K = x(\varphi_0)$$
$$\Psi_0(x) = (\mathrm{dn}\,x + e\,\mathrm{cn}\,x)^2, \quad \Psi_1(x) = \mathrm{dn}\,x(\mathrm{dn}\,x + e\,\mathrm{cn}\,x)^2, \quad \kappa = x(\varphi_1)$$

The equation of the surface then takes the simpler form

$$r = a\sqrt{\chi_0(x)}, \quad z = a(e\,\mathrm{sn}\,x + E(x))$$
$$E(x) = \int_0^x \mathrm{dn}^2 v dv, \quad -\kappa \le x \le \kappa \tag{1.1}$$

Hence, we have

$$h = z(\kappa) = a(e\,\mathrm{sn}\,\kappa + E(\kappa))$$

and an element of the surface is specified as

$$dS = a^2\Psi_0(x) d\theta dx$$

The equation of the perturbed free surface differs from (1.1) in that there is a term $\zeta(\theta, x, t)$ on the right-hand side of the first equality of (1.1). The function ζ and its derivatives are assumed to be small.

Note that the normal displacement of a point of the free surface is given, apart from a term of the first order of smallness, by $\zeta\cos\Phi = \zeta\,\mathrm{dn}\,x$ (Fig. 2).

The function $\zeta(\theta, x, t)$ must therefore satisfy the periodicity conditions

$$\zeta(\theta, x, t) = \zeta(\theta + 2\pi, x, t) \tag{1.2}$$

the boundary conditions

$$\zeta(\theta, \pm\kappa, t) = 0 \tag{1.3}$$

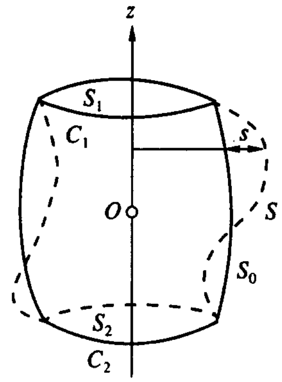

Fig. 2

and a condition which expresses the constancy of the fluid volume

$$\int_{S_0} \zeta \cos\Phi dS = 0$$

which is reducible to the form

$$\int_{\Omega} \zeta \chi_1(x) d\theta dx = 0 \tag{1.4}$$

where the domain $\Omega = \{(\theta, x)\colon 0 < \theta \leqslant 2\pi, -\kappa < x < \kappa\}$ is a rectangle.

We will assume that the fluid is ideal and incompressible; its constant density is equal to ρ and the fluid flow is irrotational. The velocity potential $\Phi(r, \theta, x, t)$ is then a harmonic function, that is

$$\Delta\Phi = 0 \tag{1.5}$$

in the domain τ occupied by the fluid, and the function Φ must satisfy the kinematic conditions.

$$\partial\Phi/\partial n = \zeta_t \operatorname{dn} x \text{ on } S_0(\zeta_t = \partial\zeta/\partial t); \quad \partial\Phi/\partial n = 0 \text{ when } z = \pm h \tag{1.6}$$

where $\partial/\partial n$ is a derivative along the outer normal to the domain τ.

Suppose p_0 is the constant external pressure and p is the pressure in the fluid. The dynamic condition on the free surface is given by Laplace's law [12]

$$p - p_0 = -\alpha(R_1^{-1} + R_2^{-1})$$

where α is the surface tension, which is assumed to be constant, and R_1 and R_2 are the radii of curvature of the perturbed free surface S, which are assumed to be negative when the centre of principal curvature is located on the same side of the surface as the fluid.

On calculating the mean curvature of the surface S using the general formula [13] in the first approximation with respect to ζ, we obtain

$$R_1^{-1} + R_2^{-1} = -a^{-1} + a^{-2}\mathfrak{D}[\zeta], \quad \mathfrak{D}[\zeta] = [(\zeta_{\theta\theta} + \zeta)\operatorname{dn}^2 x + (\zeta_x \operatorname{dn}^2 x)_x]/\Psi_1(x)$$

The linearized Bernoulli equation then gives

$$\partial\Phi/\partial t|_{S_0} = \mu\mathfrak{D}[\zeta] + C(t), \quad \mu = \alpha\rho^{-1}a^{-2} \tag{1.7}$$

where $C(t)$ is an arbitrary function of time.

Conditions (1.2)–(1.4) have to be added to Eq. (1.5) and conditions (1.6).

2. THE OPERATOR EQUATION OF THE PROBLEM

We will now consider the auxiliary Neumann problem $\Delta\Phi = 0$ in the domain τ; $\partial\Phi/\partial n = 0$ when $z = \pm h$ with the compatibility condition

$$\int_{S_0} g \,\mathrm{dn}\, x dS = 0$$

If

$$\tilde{H} = \left\{ g \in L^2(S_0), \int_{S_0} g \,\mathrm{dn}\, x dS = 0 \right\}, \quad \tilde{H}^1(\tau) = \left\{ u \in H^1(\tau), \int_{S_0} u|_{S_0} \mathrm{dn}\, x ds = 0 \right\}$$

then the unique function $\Phi \in \tilde{H}^1(\tau)$ determining the weak solution of the Neumann problem such that

$$\int_{\tau} \mathrm{grad}\,\Phi\, \mathrm{grad}\,\Psi d\tau = \int_{S_0} g\Psi|_{S_0} \mathrm{dn}\, x dS, \quad \forall \Psi \in \tilde{H}^1(\tau)$$

corresponds to every function $g \in \tilde{H}$.

The trace $\Phi|_{S_0}$ of the function Φ on the surface S_0 belongs to the space $\tilde{H}$. A linear operator $\mathbf{K}$ from $\tilde{H}$ into $\tilde{H}$ can therefore be defined as follows:

$$\Phi|_{S_0} = \mathbf{K}g$$

It is well known [14] that the operator $\mathbf{K}$ is continuous, self-adjoint, positive definite and compact from $\tilde{H}$ into $\tilde{H}$. In the case under consideration, $g = \zeta_t$ and Eq. (1.7) takes the form

$$\mathbf{K}\zeta_{tt} = \mu\mathfrak{D}[\zeta] + C(t).$$

Multiplying the left- and right-hand sides of this equation by $\chi_1(x)$, integrating the resulting equality over the domain Ω and taking account of the fact that $\zeta \in \tilde{H}$, $\mathbf{K}\zeta \in \tilde{H}$, we express $C(t)$ as a function of ζ. The operator equation then takes the form

$$\mathbf{K}\zeta_{tt} + \mu\mathbf{M}\zeta = 0 \tag{2.1}$$

where

$$\mathbf{M}\zeta = -\mathfrak{D}[\zeta] + (2\pi)^{-1}\mathcal{P}[\zeta]/\mathfrak{Q}$$

$$\mathcal{P}[\zeta] = \int_{\Omega} \zeta\, \mathrm{dn}^2 x d\theta dx + \mathrm{dn}^2\kappa \int_0^{2\Omega} [\zeta_x(\theta, \kappa, t) - \zeta_x(\theta, -\kappa, t)] d\theta$$

$$\mathfrak{Q} = \int_{-\kappa}^{\kappa} \chi_1(x) dx$$

Below, together with the space $\tilde{H}$, we introduce the space

$$H = \left\{ u \in L^2(\Omega): \int_{\Omega} u\chi_1(x) d\theta dx = 0 \right\}$$

with the scalar product

$$(u, v)_H = \int_{\Omega} uv\chi_0(x) d\theta dx$$

The operator $\mathbf{M}$ will be considered as an unbounded operator in H and the domain of definition of this operator is specified as

$$D(\mathbf{M}) = \left\{u \in H^2(\Omega)\colon u = 0 \quad \text{when} \quad x = \pm\kappa, \int_\Omega u \upsilon \chi_0(x) d\theta dx = 0\right\}$$

The zeroth and first order traces of the function i when $\theta = 0$, $\theta = 2\pi$ are considered as equal functions in $L^2(-\kappa, \kappa)$.

In order to obtain a bilinear form for u, $\upsilon \in D(\mathbf{M})$, associated with $\mathbf{M}$, it is subsequently necessary to calculate the scalar product u, $\upsilon \in D(\mathbf{M})$

$$(\mathbf{M}u, \upsilon)_H = \int_\Omega [-\operatorname{dn}^2 x(u_{\theta\theta} + u) - (u_x \operatorname{dn}^2 x)_x] \upsilon d\theta dx$$

After integration by parts, we have the form

$$m(u, \upsilon) = \int_\Omega (u_\theta \upsilon_\theta + u_x \upsilon_x - u\upsilon) \operatorname{dn}^2 x d\theta dx \tag{2.2}$$

the domain of definition of which has the form

$$V = \left\{u \in H^1(\Omega)\colon u = 0 \quad \text{for} \quad x = \pm\kappa, \int_\Omega u \chi_1(x) d\theta dx = 0\right\}$$

The traces of the function u when $\theta = 0$ and $\theta = 2\pi$ are equal in $L^2(-\kappa, \kappa)$. The form m is obviously symmetric and continuous in $V \times V$.

We will now give a mechanical interpretation of the form m. The potential energy Π of the surface tension forces is given by the formula [1]

$$\frac{d\Pi}{dt} = -\alpha \int_{S_0} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \zeta_t \operatorname{dn} x dS$$

Using the formula for the mean curvature and integrating by parts, we have

$$\Pi = \alpha m(\zeta, \zeta)$$

It is well known that an equilibrium position of a fluid bridge is stable if the quadratic form $m(\zeta, \zeta)$ is positive definite.

3. THE POSITIVE DEFINITENESS OF THE QUADRATIC FORM $m(\zeta, \zeta)$: REDUCTION TO AN EIGENVALUE PROBLEM

We now define the quantity

$$\lambda = \inf_{u \in V} \frac{\mathcal{M}[u]}{\mathcal{N}[u]}$$

$$\mathcal{M}[u] = \int_\Omega (u_\theta^2 + u_x^2) \operatorname{dn}^2 x d\theta dx, \quad \mathcal{N}[u] = \int_\Omega u^2 \operatorname{dn}^2 x d\theta dx$$

It exists and is non-negative. According to the definition of a lower bound, a sequence

$$\{u_n\} \in V\colon \mathcal{N}[u_n] = 1$$

exists such that

$$\lambda = \lim_{n \to \infty} \mathcal{M}[u_n]$$

This sequence is bounded in $H^1(\Omega)$. It is then possible to choose a subsequence from it, which we shall also denote by $\{u_n\}$, which weakly converges in $H^1(\Omega)$ and strongly converges in $L^2(\Omega)$ to a function

$U \in H^1(\Omega)$. It is proved using the Banach–Saks–Mazur theorem [15, 16] that the traces of the function U when $\theta = 0$, $\theta = 2\pi$ are equal in $L^2(-\kappa, \kappa)$ and, hence, $U \in V$. Using the same theorem, it is next proved that the lower bound λ is reached when $u \in U$ and, finally, that $\lambda > 0$.

We will assume [15, 16] that

$$u = U + \varepsilon\delta u, \quad \varepsilon \in R, \quad \delta u \in V$$

Then

$$\mathcal{M}[U + \varepsilon\delta u] - \lambda\mathcal{N}[U + \varepsilon\delta u] \geq 0, \quad \forall\varepsilon \in R, \quad \forall\delta u \in V$$

and, hence

$$\int_\Omega (U_\theta\delta u_\theta + U_x\delta u_x)\mathrm{dn}^2 x d\theta dx - \lambda\int_\Omega U\mathrm{dn}^2 x\delta u d\theta dx = 0, \quad \forall\delta u \in V$$

or, using the Lagrange multiplier μ, we have

$$\int_\Omega (U_\theta\delta\tilde{u}_\theta + U_x\delta\tilde{u}_x)\mathrm{dn}^2 x d\theta dx - \lambda\int_\Omega U\mathrm{dn}^2 x\delta\tilde{u} d\theta dx + \mu\int_\Omega \chi_1(x)\delta\tilde{u} d\theta dx = 0$$

$$\forall\delta\tilde{u} \in \tilde{V}$$

where

$$\tilde{V} = \{u \in H^1(\Omega) \colon U = 0 \quad \text{then} \quad x = \pm\kappa\}$$

and the traces of the function U for $\theta = 0$ and $\theta = 2\pi$ are equal in $L^2(-\kappa, \kappa)$. On putting $\delta\tilde{u} \in D(\Omega)$, which is permissible, we have

$$\mathcal{L}(U) \doteq U_{\theta\theta}\mathrm{dn}^2 x + (\mathrm{dn}^2 x U_x)_x + \lambda U\mathrm{dn}^2 x - \mu\chi_1(x) = 0$$

in the sense of distributions and, therefore, also in the sense of functions, since the solutions of this elliptic equation belong to C^∞ [17].

On calculating μ by multiplying this equation directing by $\chi_0(x)/\mathrm{dn}x$ and integrating over the domain Ω, we see that the function $U \in C^\infty$, which minimizes the ratio in relation (3.1), is the classical solution of the auxiliary eigenvalue problem

$$\begin{aligned}
&\mathcal{L}(u) = 0, \quad \mu = \mathcal{A}_\mu/(2\pi\mathcal{B}) \\
&u = 0 \quad \text{for} \quad x = \pm\kappa, \ \int_\Omega u\chi_1(x) d\theta dx = 0 \\
&u(\theta, x) = u(\theta + 2\pi, x) \\
&\mathcal{A}_\mu = \Psi_1(\kappa)\int_0^{2\pi} [u_x(\theta, \kappa) - u_x(\theta, -\kappa)]d\theta - \int_\Omega u_x(\Psi_{-1}(x))_x \mathrm{dn}^2 x d\theta dx \\
&\mathcal{B} = \int_{-\kappa}^{\kappa} \Psi_0^2(x) dx \\
&\Psi_{-1}(x) = (\mathrm{dn}x + e\,\mathrm{cn}x)^2/\mathrm{dn}x
\end{aligned} \tag{3.1}$$

and that λ is the smallest eigenvalue of the problem.

We next find the eigenvalues of problem (3.1) and investigate the conditions under which the smallest of these eigenvalues is strictly greater than unity, since this condition turns out to be the unique condition for stability.

4. THE EXISTENCE OF THE EIGENVALUES OF THE AUXILIARY PROBLEM

We shall seek solutions of problem (3.1) in the form

$$u = \Theta(\theta)X(x)$$

We have

$$\Theta'' X \,\mathrm{dn}^2 x + \Theta[(\mathrm{dn}^2 x X')' + \lambda X \,\mathrm{dn}^2 x] - \nu\tilde{\Theta}\chi_1(x) = 0$$

where

$$\nu = \frac{\mathcal{A}_\nu}{2\pi\mathcal{B}}, \quad \tilde{\Theta} = \int_0^{2\pi} \Theta(\theta) d\theta$$

$$\mathcal{A}_\nu = \Psi_1(\kappa)[X'(\kappa) - X'(-\kappa)] - \int_{-\kappa}^{\kappa} (\Psi_{-1}(x))_x \mathrm{dn}^2 x X'(x) dx$$

and the condition

$$\tilde{\Theta} C_1[X] = 0, \quad C_1[X] = \int_{-\kappa}^{\kappa} X(x)\chi_1(x) dx$$

is satisfied.

We shall distinguish between several cases.

In the first case

$$\tilde{\Theta} = 0$$

We have

$$-\Theta''/\Theta = [(X'\mathrm{dn}^2 x)' + \lambda X \,\mathrm{dn}^2 x]/(X \,\mathrm{dn}^2 x) = n^2, \quad n = 1, 2, \ldots$$

Then

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad A_n, B_n - \text{const}$$

and we have the classical Sturm–Liouville problem

$$(X'\mathrm{dn}^2 x)' + (\lambda - n^2) X \,\mathrm{dn}^2 x = 0, \quad n = 1, 2, \ldots, \quad X(\pm\kappa) = 0$$

On putting $y(x) = X(x)\mathrm{dn}x$, we obtain the problem

$$y'' = (2e^2 \mathrm{sn}^2 x - e^2 - \lambda + n^2) y, \quad n = 1, 2, \ldots, \quad y(\pm\kappa) = 0$$

Its variational formulation has the following form: it is required to find a function $y \in H_0^1(-\kappa, \kappa)$ such that

$$\int_{-\kappa}^{\kappa} [y'\tilde{y}' + (ie^2 \mathrm{sn}^2 x - e^2) y\tilde{y}] dx = (\lambda - n^2) \int_{-\kappa}^{\kappa} y\tilde{y} dx, \quad \forall \tilde{y} \in H_0^1(-\kappa, \kappa)$$

The injection from $H_0^1(-\kappa, \kappa)$ into $L^2(-\kappa, \kappa)$ is continuous, dense and compact. On the other hand, the bilinear form of the left-hand side of the last equality, which is equal to $\int_{-\kappa}^{\kappa} \mathrm{dn}^2 x dx$, is symmetric, continuous and coercive in $H_0^1(-\kappa, \kappa)$. The problems being considered are none other than standard eigenvalue problems. The eigenvalues are strictly greater than n^2 and, thereby, unity. For each value $n = 1, 2, \ldots$, the functions $y_{nm}(x)$ $(m = 1, 2, \ldots)$ form a complete orthogonal system in $L^2(-\kappa, \kappa)$.

In the second case,

$$C_1(x) = 0$$

In this case, the function $\Theta(\theta)$ is constant and, on again putting $y(x) = X(x)\mathrm{dn}x$, we have

$$y'' = (2e^2 \mathrm{sn}^2 x - \lambda - e^2) y + \nu(\mathrm{dn}x + e\,\mathrm{sn}x)^2 = 0, \quad y(\pm\kappa) = 0$$
$$\mathscr{C}[y] = \int_{-\kappa}^{\kappa} y(x)(\mathrm{dn}x + e\,\mathrm{sn}x)^2 dx = 0 \tag{4.1}$$

We introduce the spaces

$$\tilde{\mathcal{H}} = \{y \in L^2(-\kappa, \kappa): \mathcal{C}[y] = 0\}, \quad \tilde{\mathcal{V}} = \{y \in H_0^2(-\kappa, \kappa): \mathcal{C}[y] = 0\}$$

It can be shown that, as earlier, this here takes the form of a standard eigenvalue problem. These eigenvalues are strictly positive and their eigenfunctions $y_{0m}(x)$ ($m = 1, 2, \ldots$) form a complete orthogonal system in $\tilde{\mathcal{H}}$. It is well known that 1, $\cos n\theta$, $\sin n\theta$ ($n = 1, 2, \ldots$) form a complete orthogonal system in $L^2(0, 2\pi)$. Then, using the classical theorem in [18], we see that the functions

$$y_{0m}(x), \quad y_{nm}(x)\cos n\theta, \quad y_{nm}(x)\sin n\theta, \quad n, m = 1, 2, \ldots$$

form a complete orthogonal system in $L^2(\Omega)$, the elements of which satisfy the equality

$$\int_\Omega u\chi_1(x)d\theta dx = 0$$

The method of separation therefore ensures that all of the eigenvalues of problem (3.1) are found.

Hence, it is necessary to find the sufficient conditions such that all the eigenvalues of problem (4.1) are strictly greater than unity.

5. THE TRANSFORMATION OF PROBLEM (4.1)

Direct integration of differential equations (4.1) is not simple. Note that the smallest eigenvalue of problem (4.1) has the form

$$\lambda_1 = \inf_{y \in \tilde{V}} \frac{\mathcal{J}_1}{\mathcal{J}_2}, \quad \mathcal{J}_1 = \int_{-\kappa}^{\kappa} [y'^2 + e^2(2\operatorname{sn}^2 x - 1)y^2]dx, \quad \mathcal{J}_2 = \int_{-\kappa}^{\kappa} y^2 dx$$

We now consider the auxiliary problem which is obtained if one puts $\nu = 0$ in problem (4.1). It has the form

$$-y'' + e^2(2\operatorname{sn}^2 x - 1)y = \lambda' y, \quad y(\pm\kappa) = 0 \tag{5.1}$$

Its eigenvalues are strictly positive.

Lemma 1. The quantity $\lambda' = 1$ is not an eigenvalue.

Proof. In fact, when $\lambda' = 1$, the differential equation admits of a particular solution $y = \operatorname{sn} x$ and is found to be integrable. We have

$$y = AF(x) + B\operatorname{sn} x, \quad F(x) = (x - E(x))\operatorname{sn} x - \operatorname{cn} x \operatorname{dn} x, \quad A, B - \text{const}$$

The boundary conditions $y(\pm\kappa) = 0$ give

$$B = 0, \quad AF(\kappa) = 0$$

It is easily verified than, in the segment $[0, K]$ the function $F(x)$ increases from -1 to $K - E(K) > 0$ and have it vanishes exactly once. The quantity $F(\kappa)$ is then always non-zero with the exception of a single value of κ, denoted by φ_1, which we discard. Then, $A = 0$ which completes the proof.

Lemma 2. The quantity $\lambda_1 \neq 1$.

Proof. To prove this, we will show that, when $\lambda = 1$, problem (4.1) does not have solutions other than the trivial solution $y = 0$. In fact, in this case the differential equation has the form

$$[y'\operatorname{sn} x - y\operatorname{sn}' x]' = -\frac{\nu}{2e}[(2\operatorname{sn}^2 x - 1)]'$$

It is integrated and its general solution has the form

$$y = \nu(1 + ex\operatorname{sn} x) + B\operatorname{sn} x + AF(x)$$

From the conditions of problem (4.1), we find

$$B = 0, \quad AF(\kappa) + \nu(1 + e\kappa \operatorname{sn}\kappa) = 0$$

$$A\mathcal{I}_3 + \nu\mathcal{I}_4 = 0, \quad \mathcal{I}_3 = \int_0^\kappa F(x)(2\operatorname{sn}^2 x - 1)dx$$

$$\mathcal{I}_4 = \int_0^\kappa (1 + ex\operatorname{sn}x)(2\operatorname{sn}^2 x - 1)dx$$

By integrating by parts, it can be shown that the coefficient of A in the last condition is negative and it follows from this that, if both ν and A are non-zero, the ratio $\nu_A = \nu/A$ must be positive. It is easily shown that, in this case

$$\mathcal{I}_3 + \nu_A\mathcal{I}_4 < (1 + e^2)\kappa[F(\kappa) + \nu_A(1 + e\kappa\operatorname{sn}\kappa)] = 0$$

We therefore arrive at a contradiction, the quantities ν and A are equal to zero, whence $y = 0$.

Lemma 3. Problem (5.1) admits of a single eigenvalue which is strictly less than unity while the remaining eigenvalues are greater than unity.

Proof. The problem

$$-y'' + e^2(2\operatorname{sn}^2 x - 1)y = \lambda'' y, \quad y(\pm\kappa) = 0$$

admits of one and just one negative eigenvalue while all the remaining eigenvalues are greater than unity. In this case, we can make use of the following theorem [7].

Suppose Φ_0 is the solution of the problem

$$-y'' + e^2(2\operatorname{sn}^2 x - 1)y - y = f(x), \quad y(\pm\kappa) = 0$$

If

$$\int_{-\kappa}^{\kappa} \Phi_0 f dx < 0$$

then the quadratic form $\mathcal{I}_4 = \mathcal{I}_1 - \mathcal{I}_2$ is positive definite in $\tilde{\mathcal{V}}$. As will be shown below, this condition is satisfied when $f(x) = \chi_0(x)$. Under these conditions, we have

$$\inf_{y \in \tilde{\mathcal{V}}} (\mathcal{I}_1 - \mathcal{I}_2)/\mathcal{I}_2 \geq 0$$

and, as a corollary, we have $\lambda_1 > 1$ by virtue of the fact that $\lambda_1 \neq 1$.

Henceforth, we will confine ourselves to searching for the sufficient conditions in order that problem (5.1) should have one and just one eigenvalue which strictly exceeds unity.

We put

$$\lambda' = 1 - e^2\operatorname{sn}^2\omega$$

Then, the differential equation of problem (5.1) takes the classical form of Lamé's equation

$$y'' = (2e^2\operatorname{sn}^2 x - 1 - e^2 + e^2\operatorname{sn}^2\omega)y$$

and we subsequently make use of the classical concepts of the theory of elliptic functions from [19, 11]. We denote the real and pure imaginary parts of the half-periods of the function $\wp u$ by ω_1 and ω_3 respectively and the zeros of the function $\wp' u$ by e_1, e_2 and e_3 where $e_1 > e_2 > e_3$. Finally, we suppose that $\eta_1 = \zeta\omega_1$, $\eta_3 = \zeta\omega_3$.

We assume that the eigenvalues $\lambda' \in (0, 1)$ and find where the corresponding value of the quantity ω must be located. We have

$$0 < \operatorname{sn}^2\omega < e^{-2}$$

However, we know [11] that

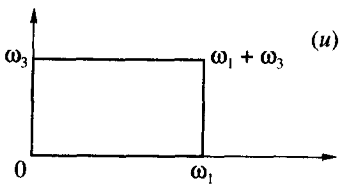

Fig. 3

$$\operatorname{sn}^2\omega = (e_1-e_3)/(\wp u-e_3), \quad u = \omega_1\omega/K$$

and the function $\wp u$ takes real values in a rectangular contour with vertices 0, ω_1, $\omega_1+\omega_3$ and ω_3 (Fig. 3). It decreases monotonically from $+\infty$ to $-\infty$ when u runs through the contour in a clockwise direction starting from zero.

The double inequality

$$0<(e_1-e_3)/(\wp u-e_3)<e^{-2}$$

is equivalent to the inequality $\wp u > e_2$. It follows from this that the condition $0<\lambda'<1$ is equivalent to the inequality $0<u<\omega_1$ and, therefore, $\wp u > e_1$. For $u=\omega_1+iy_0$, $0<y_0<\omega_3$, this condition is equivalent to the inequality $e_2<\wp u<e_1$. In the first case, we have $1-e^2<\lambda'<1$ and, in the second case, $-0<\lambda'<1-e^2$.

Furthermore, Hermite [20] found the general solution of Lamé's equation in the form

$$y = A\theta_1\left(\frac{x+\omega}{2K}\right)\theta_4^{-1}\left(\frac{x}{2K}\right)\exp\left(-\mathcal{F}\left(\frac{\omega}{2K}\right)\frac{x}{2K}\right)+B\theta_1\left(\frac{-x+\omega}{2K}\right)\theta_4^{-1}\left(\frac{x}{2K}\right)\exp\left(\mathcal{F}\left(\frac{\omega}{2K}\right)\frac{x}{2K}\right)$$

$$\mathcal{F}(x) = \frac{\theta_4'(x)}{\theta_4(x)}$$

if both particular integrals, that is, the coefficients of A and B, are linearly independent. Henceforth, θ_α is Jacobi's theta-function [19].

We will now investigate the problem of the existence of values of ω, corresponding to $0<\lambda'<1$, which are such that the two particular integrals are linearly independent. We note that $\theta_4(\omega/(2K))=0$ if $u=\omega_3+2n\omega_1+2m\omega_3$, where n and m are integers. Hence, the function $\theta_4(\omega/(2K))$ cannot vanish when $0<u<\omega_1$, $u\in R$ and when $u=\omega_1+iy_0$, $0<y_0<\omega_3/i$.

We will now write the conditions that the right-hand side of equality (5.2) should vanish when $x=0$ and $x=iK$. We have

$$\theta_1\left(\frac{\omega}{2K}\right)(A+B) = 0, \quad \theta_1\left(\frac{\omega}{2K}\right)\left(A\exp\left(-\mathcal{F}\left(\frac{\omega}{2K}\right)\right)+B\exp\left(\mathcal{F}\left(\frac{\omega}{2K}\right)\right)\right) = 0$$

The case when $\theta_1(\omega/(2K))=0$ has to be discarded since, then, $\omega=0$ and $\lambda'=1$. Hence A and B are non-zero if

$$\mathcal{F}(\omega/(2K)) = k\pi i, \quad k\in Z$$

where [19]

$$\zeta_3 u-\eta_1 u/\omega_1 = k\pi i/(2\omega_1)$$

For $0<u<\omega_1$, $u\in R$, it is necessary to put $k=0$.

The function $g(u)=\zeta_3 u-\eta_1 u/\omega_1$ is non-zero when $u=0$ and $u=\omega_1$, and its derivative $g'u = -\wp(u+\omega_3)-\eta_1/\omega_1$ is positive when $u=0$ and vanishes once in the interval $(0,\omega_1)$.

The function $g(u)$ therefore vanishes when $u=0$, that is, when $\lambda'=1$ which is excluded from the treatment and when $u=\omega_1$, that is, when $\lambda'=1-e^2$.

For $u=\omega_1+iy_0$, $0<y_0<\omega_3/i$, the equation has the form

$$\zeta_2(iy_0)/i-\eta_1 y_0/\omega_1 = k\pi/(2\omega_1)$$

It can be shown that the derivative of the function on the left-hand side, which is equal to $-\wp(\omega_2 + iy_0) - \eta_1/\omega_1$, is negative. The function $\zeta_2(iy_0)/i - \eta_1 y_0/\omega_1$ decreases from zero to $-\pi/(2\omega_1)$ in the interval $(0, \omega_3/i)$, from which it follows that the equation cannot be satisfied except when $k = 0$ and $k = -1$. If $k = 0$, we have $y_0 = 0$, and hence $u = \omega_1$, $\omega = K$ and $\lambda' = 1 - e^2$. If $k = -1$, we have $y_0 = \omega_3/i$ and hence $u = \omega_1 + \omega_3$, $\text{sn}\omega = e^{-1}$ and $\lambda' = 0$, which is excluded from the treatment.

We will now show that $\lambda' = 1 - e^2$ cannot be an eigenvalue of problem (5.1). The problem has the form

$$y'' = (2\text{sn}^2 x - 1)y, \quad y(\pm\kappa) = 0$$

The differential equation possesses a particular solution $\text{cn}x$, it can be integrated and its general solution has the form

$$y = A\,\text{cn}x G(x) + B\,\text{cn}x$$

$$G(x) = x - (1-e^2)^{-1}E(x) + (1-e^2)^{-1}\text{sn}x\,\text{dn}x\,\text{cn}^{-1}x$$

The conditions $y(\pm\kappa) = 0$ give

$$B = 0, \quad AG(\kappa) = 0$$

It is easy to show that the function $G(x)$, the derivative of which is equal to $\text{cn}^{-2}x$, is positive in the interval $[0, \kappa]$, and it follows this that $A = 0$. This completes the proof.

So, for values of ω corresponding to $\lambda' \in (0, 1)$, the two particular integrals of Lamé's equation are linearly independent.

It is now necessary to find the eigenvalues of problem (5.1) for those values of ω for which the eigenvalues are strictly less than unity. We shall seek the sufficient conditions for which there is just one such eigenvalue.

6. THE SUFFICIENT CONDITION FOR THE EXISTENCE IN PROBLEM (5.1) OF A SINGLE EIGENVALUE WHICH IS STRICTLY LESS THAN UNITY

It is necessary to consider the two cases mentioned below.

First case: $0 < u < \omega_1$. In this case $\omega \in R$ and $0 < \omega < K$. From the general integral (5.2) of Lamé's equation, we find that $\lambda' = 1 - e^2\text{sn}^2\omega$ is an eigenvalue for real values of ω, which satisfy the equation

$$\theta_1\left(\frac{\omega-\kappa}{2K}\right)\theta_1^{-1}\left(\frac{\omega-\kappa}{2K}\right) = \pm\exp\left(-\mathcal{F}\left(\frac{\omega}{2K}\right)\kappa K^{-1}\right) \tag{6.1}$$

Using the expression for $\theta(\omega/(2K))$ found above and the formula [19]

$$\theta_1(\upsilon) = \exp(-2\eta_1\omega_1\upsilon)\frac{\pi}{\omega_1}q_0^2 q^{1/4}\sigma(2\omega_1\upsilon), \quad q = \exp\left(\frac{i\pi\omega_3}{\omega_1}\right), \quad q_0 = \prod_{n=1}^{\infty}(1-q^{2n})$$

we represent Eq. (6.1) in the form

$$\tilde{\omega}u = \frac{\sigma(u-u_1)}{\sigma(u+u_1)} = \pm\exp(-2u_1\zeta_3 u), \quad 0 < u_1 = \frac{\omega_1\kappa}{K} < \omega_1 \tag{6.2}$$

This equation can be solved graphically. We will investigate the left-hand side of the equation. We have

$$[\tilde{\omega}(u)]' = \frac{\sigma(u-u_1)}{\sigma(u+u_1)}F_0(u), \quad F_0(u) = -\zeta u_1 + \frac{\wp' u}{\wp u - \wp u_1}$$

We have

$$F_0(0) = 0, \quad F_0(u_1) = \infty, \quad F_0(\infty) = -2\zeta_1 u_1$$

Since $\zeta_1 u$ decreases from zero to $-\infty$ in the interval $(0, \omega_1)$, we have $F_0(\omega_1) > 0$. On the other hand, it can be verified that $F_0'(u) < 0$.

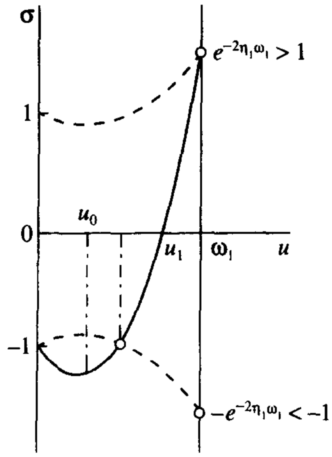

Fig. 4

It is clear from the graph of the function $F_0(u)$ that, if $\zeta u_1 < 0$, the function $F_0(u)$ has a single zero u_0 and $0 < u_0 < u_1$ but, if $\zeta u_1 > 0$, the function $F_0(u)$ does not vanish in the interval $(0, \omega_1)$. The following result can be derived from this: if $\zeta u_1 < 0$ (and, consequently, $\eta_1 < 0$ and ζu has a single zero β in the interval $(0, \omega_1)$), the function $\tilde{\sigma}(u)$ decreases from -1 in the interval $(0, u_0)$ and increases up to $\exp(-2\eta_1\omega_1) > 1$ in the interval (u_0, ω_1). If $\zeta u_1 > 0$, then this function increases from -1 up to $\exp(-2\eta_1\omega_1)$ in the interval $(0, \omega_1)$.

We now investigate the function $\pm\exp(-2u_1\zeta_3 u)$. We have [19]

$$\zeta_3' u = -\wp(u+\omega_3) - e_3 - (e_3 - e_1)(e_3 - e_2)/(\wp u - e_3)$$

We conclude from this that, if $e_2 < 0$, the function ζ_3' does not vanish in the interval $(0, \omega_1)$ and that, if $e_2 > 0$, this function can have a single zero γ in the interval $(0, \omega_1)$. Then, if $e_2 < 0$, the function $\exp(-2u_1\zeta_3 u)$ decreases from unity to $\exp(-2\eta_1\omega_1)$ and, if $e_2 > 0$, this function has a minimum when $u = \gamma$.

We can investigate Eq. (6.2) graphically in this way. In the case when $\zeta u_1 < 0$, which implies that $\eta_1 < 0$ and $e_2 > 0$, we have in Fig. 4 a graph of the function $\tilde{\omega}(u)$ depicted by the solid curve and plots of the functions $\pm\exp(-2u_1\zeta_3 u)$ depicted by the dashed curves.

Hence, Eq. (6.2) has a single root belonging to the interval $(0, u_1)$.

In the case when $\zeta u_1 > 0$, on comparing the angular coefficients of the tangents to the curves at the point $u = 0$, we conclude that Eq. (6.2) has a single root in the interval $(0, u_1)$ if the condition $\zeta u_1 + e_3 u_1 < 0$ is satisfied. Moreover, in the case of a real u which satisfies the inequality $0 < u < \omega_1$, Eq. (6.2) has a single root if $\zeta u_1 < 0$ or if $0 < \zeta u_1 - e_3 u_1$.

Second case: $u = \omega_1 + iy_0$, $0 < y_0 < \omega_3/i$. In this case $\lambda' = 1 - e^2\mathrm{sn}^2\omega$ is an eigenvalue for the values $u\omega_1\omega/K$ of Eqs (6.2). Using the relations between the functions σ and σ_1 [19], we find the equivalent equation in y

$$\tilde{\sigma}_1(iy_0) = \frac{\sigma_1(iy_0 - u_1)}{\sigma_1(iy_0 + u_1)} = \pm\exp(-2u_1\zeta_2(iy_0)) \tag{6.3}$$

It can be shown at once that the quantities on the left- and right-hand sides have a modulus equal to unity. We now replace (6.3) with the equation

$$\tilde{\sigma}(iy_0)^2 = \pm\exp(-4u_1\zeta_2(iy_0)) \tag{6.4}$$

When $y_0 = 0$ ($y_0 = \omega_3/i$ respectively), both sides of the equation are equal to unity (to $\exp(-4\eta_3 u_1)$ respectively). Hence, $y_0 = 0$ and $y_0 = \omega_3/i$ are solutions of the equation, from which it follows that $u = \omega_1$ and $u = \omega_1 + \omega_3$ are solutions of Eq. (6.2).

We will assume that the argument on each of the sides is equal to zero and trace the values of the argument when y_0 increases from zero to ω_3/i.

We have [19]

$$\frac{d}{dy_0}\left[\arg\left(\frac{\sigma_1(iy_0-u_1)}{\sigma_1(iy_0+u_1)}\right)^2\right] = 2[\zeta_1(iy_0-u_1)-\zeta_1(iy_0+u_1)] =$$

$$= 2\left[-2\zeta u_1 + \frac{\wp' u_1}{\wp(\omega_1+iy_0)-\wp u_1}\right]$$

But, here, we have

$$e_2 < \wp(\omega_1+iy_0) < e_1 < \wp u_1 \quad \text{and} \quad \wp' u_1 < 0$$

whence it follows that

$$\frac{\wp' u_1}{\wp(\omega_1+iy_0)-\wp u_1} > 0$$

If $\zeta u_1 > 0$, the derivative being considered can vanish and a full discussion of this is extremely complex. We shall discuss the case when $\zeta u_1 < 0$. In this case, the derivative under consideration is positive and $\tilde{\sigma}(iy_0)^2$ constantly increases from zero in the interval $(0, \omega_3/i)$. For $\zeta u_1 < 0$, we have $\eta_1 < 0$ and $e_2 > 0$. From the known relation $\eta_1\omega_3 - \eta_3\omega_1 = i\pi/2$, we then derive $-4\eta_3\omega_1/i > 2\pi$. The argument of the function $\exp(-4\eta_3 u_1)$ then has the form $-4\eta_3 u_1 + 2N\pi$, where the integer N is chosen in such a way that this argument does not exceed 2π.

The argument of the function $\exp(-4u_1\zeta_2(iy_0))$ is equal to $-4u_1\zeta_2(iy)/i \bmod(2\pi)$. We have

$$d/dy_0[\zeta_2(iy_0)/i] = -\wp(\omega_2+iy_0) = -e_2-(e_2-e_1)(e_2-e_3)/(\wp(iy_0)-e_1)$$

The second term is negative and it follows from this that the argument of the function $\exp(-4u_1\zeta_2(iy_0))$ increases from zero to $-4\eta_3 u_1/i - 2N\pi$.

Finally, it can be shown that, in the interval $(0, \omega_3/i)$, the second derivatives of the functions $\arg[\tilde{\sigma}_1(iy_0)^2]$ and $-4u_1\zeta_2(iy_0)/i$ are negative and positive respectively. The graph of the first function then lies above the graph of the second function, which proves the existence in Eq. (6.4) of just the two roots $y_0 = 0$ and $y_0 = \omega_3/i$. Equation (6.2) only possesses two roots from which it follows that $\operatorname{sn}\omega_1 = 1$ and $\operatorname{sn}\omega = e^{-1}$, that is, $\lambda' = 1 - e^2$ and $\lambda' = 0$. Since these two values have to be discarded, we obtain the following result: for $\zeta u_1 < 0$, a value of ω of the form $\omega_1 + iy_0$, $0 \leqslant y_0 \leqslant \omega_3/i$ does not exist, which ensures the existence of an eigenvalue of problem (5.1), which is strictly less than unity.

So, if $\zeta u_1 < 0$, just one real value of ω exists which corresponds to an eigenvalue $\lambda' < 1$. The fact that the inequality $\zeta u_1 < 0$ is the condition which is sufficient in order that the subsidiary problem (5.1) should have exactly one eigenvalue, which is strictly less than unity, is also substantiated.

We will now make a few remarks concerning the condition $\zeta u_1 < 0$. This condition implies the inequality $\eta_1 < 0$. From the relation [11]

$$\eta_1 = \sqrt{e_1-e_3}\left(\frac{E-Ke_1}{e_1-e_3}\right), \quad E = \int_0^{\pi/2}\sqrt{1-e^2\sin^2 \upsilon}\,d\upsilon, \quad e^2 = \frac{e_2-e_3}{e_1-e_3}$$

it is possible to derive that the conditions $\eta_1 < 0$ is equivalent to the inequality

$$K > 3E/(2-e^2)$$

We have $K \geqslant E$ and, when $e \to 1$, we have $E \to 1$ and $K \to +\infty$. Hence, the condition is satisfied when e is sufficiently close to unity.

On the other hand, the inequality $\zeta u_1 < 0$ holds if $\beta < u_1 < \omega_1$, where β is a zero of the function ζu in the interval $(0, \omega_1)$. This last condition is certainly ensured if the value of κ is sufficiently close to K or if φ_1 is sufficiently close to φ_0. The quantity

$$F(\kappa) = [\kappa - E(\kappa)]\operatorname{sn}\kappa - \operatorname{cn}\kappa\operatorname{dn}\kappa$$

is then positive, as will be assumed henceforth.

7. SUFFICIENT CONDITIONS IN ORDER THAT ALL THE EIGENVALUES OF PROBLEM (4.1) ARE STRICTLY GREATER THAN UNITY

We assume that $\zeta u_1 < 0$ and $F(\kappa) > 0$. We use Vogel's theorem, which has been mentioned above, and consider the problem

$$-y'' + e^2(2\mathrm{sn}^2 x - 1)y - y = \chi_0(x), \quad y(\pm\kappa) = 0 \tag{7.1}$$

We introduce the functional

$$G[y] = \int_{-\kappa}^{\kappa} y\chi_0(x)dx < 0$$

If Φ_0 is the solution of this problem and $G[\Phi_0] < 0$, then the quadratic functional

$$\int_{-\kappa}^{\kappa} dx[y'^2 + e^2(2\mathrm{sn}^2 x - 1)y^2 - y^2]dx$$

is positive definite in the space

$$\tilde{\upsilon} = \{y \in H_0^1(-\kappa, \kappa), G[y] = 0\}$$

and it follows from this that the smallest eigenvalue of problem (4.1)

$$\lambda_1 = \inf_{y \in \tilde{\mathcal{V}}} \mathcal{I}_1/\mathcal{I}_2$$

is strictly greater than unity.

We will now calculate the solution Φ_0. The general solution of the differential equation in problem (7.1) has the form

$$y = Y(x) + B\,\mathrm{sn}\,x, \quad Y(x) = A[(x - E(x))\mathrm{sn}\,x - \mathrm{cn}\,x\,\mathrm{dn}\,x] - (1 + ex\,\mathrm{sn}\,x)$$

By virtue of the boundary conditions, we have $B = 0$ and

$$y = Y(\kappa)$$

whence, by virtue of the fast that the coefficient of A is positive, we have $A > 0$. The Vogel condition has the form

$$G[Y(x)] < 0$$

It is satisfied since, according to what has been proved above, the coefficient of A is negative.

8. THE STABILITY OF A FLUID BRIDGE

We shall always assume that the conditions

$$\zeta u_1 < 0, \quad F(\kappa) > 0 \tag{8.1}$$

are satisfied.

By virtue of the definition of the quantity λ_1, we have

$$\mathcal{M} \geq \lambda_1 \mathcal{N}, \quad \forall u \in V$$

Suppose $0 < \varepsilon < 1$ and

$$m(u, u) = \varepsilon\mathcal{M} + (1 - \varepsilon)\mathcal{M} - \mathcal{N}$$

Then

$$m(u, u) \geq \varepsilon\mathcal{M} + [(1-\varepsilon)\lambda_1 - 1]\mathcal{N}$$

Since $\lambda_1 > 1$, it is possible to satisfy the inequality $(1-\varepsilon)\lambda_1 - 1$ by choosing a parameter ε which satisfies the inequality $0 < \varepsilon < (\lambda_1 - 1)/\lambda_1$. Since $\mathrm{dn}^2 x$ takes values between two strictly positive numbers in the interval $(-\kappa, \kappa)$, a $\tilde{\varepsilon} > 0$ can be found such that

$$m(u, u) \geq \tilde{\varepsilon}\|u\|^2_{H^1(\Omega)}, \quad \forall u \in V$$

from which the coercivity of the bilinear form m in $V \times V$ follows. Consequently, inequalities (8.1) are sufficient conditions for the stability of a wave-like bridge.

9. THE EXISTENCE OF NATURAL FREQUENCIES

From the operator equation (2.1), we derive

$$(K\zeta_{tt}, \tilde{\zeta})_H + \mu m(\zeta, \tilde{\zeta}) = 0, \quad \forall\tilde{\zeta} \in V, \quad \mu = \alpha/(\rho a^2)$$

We introduce the space $\hat{H}$, which is an enlargement of V with a norm associated with the scalar product $(u, \upsilon)_{\hat{H}} = (Ku, \upsilon)_H$.The variational formulation then takes the following form: it is required to find $\zeta(t) \in V$ such that

$$(\zeta_{tt}, \tilde{\zeta})_{\hat{H}} + \mu m(\zeta, \tilde{\zeta}) = 0, \quad \forall\tilde{\zeta} \in V \tag{9.1}$$

It is well known that, when the conditions $\zeta u_1 < 0$ and $F(\kappa) > 0$ are satisfied, the bilinear form m is symmetric, continuous and coercive in $V \times V$. On the other hand, the injection of V into $\hat{H}$ is dense by construction, continuous and compact. The last two properties immediately imply the classical properties of injection form $H^1(\Omega)$ to $L^2(\Omega)$ and the continuity of the operator K.

We therefore have a standard problem in the theory of oscillations [21]: an innumerable set of eigenvalues

$$0 < \Omega_1 \leq \Omega_2 \leq \ldots \leq \Omega_n, \quad \Omega_n \to \infty$$

exists such that the corresponding eigenfunctions form an orthogonal basis in the space $\tilde{H}$ as well as in the space V, which is equipped with the scalar product $m(.,.)$.

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